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A CHARACTERIZATION OF A GEOMETRY RELATED TO $\Omega^{+}_{2n}(K)$

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A characterization of a Geometry Related to $\Omega^+_{2n}(K)$ *)

by

Bruce N. Cooperstein **)

ABSTRACT

The halved dual polar spaces related to $\Omega^+_{2n}(K)$ are characterized as incidence structures in terms of a short list of axioms on points and lines.

KEY WORDS & PHRASES: *graphs, incidence structures, (dual) polar spaces, buildings of type* D_n.

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0. INTRODUCTION

Let V be a vector space of dimension $2n \geq 8$ over a field K equipped with a non-degenerate quadratic form Q with maximal Witt index (so totally singular subspaces of dimension n exist). Let $\mathcal{M}$ denote the collection of maximal totally singular subspaces of V. If we define the relation $x \approx y$, for $x,y \in \mathcal{M}$, if and only if $\dim_K x/x \cap y$ is even, then it is well-known that $\approx$ is an equivalence relation with two equivalence classes. Let P denote one of these classes. Let L be the collection of totally singular subspaces of V with linear dimension n-2. Then $(P, L, \subseteq \cup \supseteq)$ is an incidence structure known as $D_{n,max}(K)$ or $D_{n,n}(K)$. The purpose of this paper is to characterize these incidence structures. This extends part of Theorem B of [4]. As an application of our results, in sections 5 and 6 we obtain another proof of Cameron's characterization of the dual polar spaces of type D_n.

1. DEFINITION AND NOTATION

(1.1) DEFINITION. By *an incidence structure* here we will mean a pair of disjoint sets P and L whose members we call *points* and *lines* respectively, together with a symetric relation between them, such that each line is incident with at least two points. If every line is incident with at least three points then we say (P,L) is *thick*.

(1.2) DEFINITION. An incidence structure (P,L;I) is a *partial linear space* (pls) if two points lie on at most one line.

When (P,L;I) is a pls then no two lines are incident with the same points. Then we may identify a line with the points it is incident to and replace I with symmetrized inclusion. We will do this throughout this paper, and drop the relation I.

(1.3) DEFINITION. The *point-graph* of (P,L) is the graph (P,Γ) with vertex set P and edge set consisting of pairs of points which are collinear.

(1.4) NOTATION. If (P,Γ) is the point-graph of (P,L), then $x^{\perp} = \{x\} \cup \{y : \{x,y\} \in \Gamma\}$.
If $X \subseteq P$, $X^{\perp} = \bigcap_{x \in X} x^{\perp}$, and $\mathrm{Rad}(X) = X \cap X^{\perp}$.

(1.5) DEFINITION. If (P,Γ) is a graph and $x,y \in P$, then a *path of of length* n from x to y is a sequence $x = x_0, x_1, \ldots, x_n = y$ with $\{x_i, x_{i+1}\} \in \Gamma$ for $i = 0,1,\ldots,n-1$. If such a path exists, then the *distance* from x to y, denoted $d(x,y)$, is the length of the shortest path from x to y (such a path is called a *geodesic* or *g* - path). If no path connects x and y, then we write $d(x,y) = +\infty$.

(P,L) is *connected* if for each pair $x,y \in P$, $d(x,y) < \infty$, and in this case diam $(P,\Gamma) = \sup\{d(x,y): x,y \in P\}$. If $X,Y \subseteq P$, then $d(X,Y) = \min\{d(x,y): x \in X,\ y \in Y\}$.

(1.6) NOTATION. If (P,Γ) is a graph, $x \in P$, then $\Gamma_k(x) = \{y \in P: d(x,y) = k\}$. In [10] D.G. Higman introduced the notion of a *gamma space*. This notion is generalized in [3] to

(1.7) DEFINITION. An incidence structure (P,L) with point graph (P,Γ) is a *strong gamma space* if whenever $x \in P$, $\ell \in L$ with $d(x,\ell) = k$, then either $\ell \subseteq \Gamma_k(x)$ or $|\ell \cap \Gamma_k(x)| = 1$.

(1.8) DEFINITION. (P,L) an incidence structure with point-graph (P,Γ). A subset X of P is a *subspace* if whenever a line ℓ m-ets X in a least two points, then ℓ is contained in X. X is a *singular subspace* if X is a clique. The *rank* of a singular subspace X, denote rk(x), is defined to be the length of a maximal chain of properly ascending subspaces. For example the rank of a point os 0, of a line 1. We will call singular subspaces of rank two *planes*. By convention the empty set has rank -1.

(1.9) NOTATION. If (P,L) is an incidence structure, and $\mathcal{K}$ some collection of subspaces, and $X \subseteq P$, then $\mathcal{K}_X = \{K \in \mathcal{K} : X \subseteq K\}$ and $\mathcal{K}(X) = \{K \in \mathcal{K}: K \subseteq X\}$. We denote the collection of all subspaces of by $\underline{\underline{Sub}}$, planes by $\underline{\underline{V}}$, and singular subspaces by $\underline{\underline{Sing}}$.

(1.10) DEFINITION. For $X \subseteq P$, $\langle X\rangle$ will denote the *subspace spanned by* X, $\langle X\rangle = \cup_{S \in \underline{\underline{Sub}}_X} S$.

(1.11) DEFINITION. A *polar space* is an incidence structure (P,L) such that for any point-line pair x,ℓ, either X is collinear with one or all points of ℓ [alternatively a (strong) gamma space in which $d(x,\ell) \leq 1$]. The polar space is non-degenerate if $\mathrm{Rad}(P) = \emptyset$. The theorems of BUEKENHOUT and SHULT, [1], TITS [5] and VELDKAMP [7] classify the non degenerate polar spaces all of whose singular subspaces have finite rank. Then $\mathrm{rk}(P,L) = \max\{\mathrm{rk}\, M : M \in \underline{\underline{\mathrm{Sing}}}\} + 1$.

It is our goal in this paper to characterize incidence structures (P,L) with point graph (P,Γ) which satisfy the following axioms

(D1) (P,L) is thick and connected, (P,Γ) is not complete;

(D2) For $d(x,y) = 2, (\{x,y\}^{\perp}, L(\{x,y\}^{\perp}))$ is a thick non-degenerate polar space of rank three. If x,ℓ is a point line pair with $\ell \subseteq \Gamma_2(x)$, then $x^{\perp} \cap \ell^{\perp}$ is a singular subspace maximal in $\{x,y\}^{\perp}$ for each $y \in \ell$.

(D3) (P,L) is a strong gamma space. If $\ell \subseteq \Gamma_k(x)$ with $k \geq 3$, then $\emptyset \neq \ell^{\perp} \cap \Gamma_{k-1}(x) \in \underline{\underline{\mathrm{Sing}}}$.

We now describe the typical example:

Let V be a vector space of dimension $2n \geq 8$ ever a field K and Q a non-degenerate quadratic form on V with maximal with index (i.e. so that there exists subspaces U of dimension n with $Q(U) = \{0\}$). Let M be the collection of such subspaces. Define $U_1 \approx U_2$, for $U_1,U_2 \in M$ if $\dim U_i/U_1 \cap U_2$ is even. Then it is well known that $\approx$ is an equivalence relation with two equivalence classes. Let P be either of these classes. We will define a set of lines on P : for $U_1,U_2 \in P$ we define U_1 and U_2 to be collinear if $\dim U_i/U_1 \cap U_2 = 2$ and then $\ell(U_1,U_2) = \{U \in P : U \supseteq U_1 \cap U_2\}$. Define $L = \{\ell(U_1,U_2) : U_1,U_2$ collinear$\}$. Then we denote (P,L) by $D_{n,n}(K)$.

In [4] it is remarked that $D_{n,n}(K)$ arises as a Lie incidence structure and satisfies (D1) and (D2). By [3] it follows that $D_{n,n}(K)$ is a strong gamma space, we next prove

(1.13) <u>PROPOSITION</u>. (P,L) *satisfies* (D3).

<u>PROOF</u>. Let $\ell \in L$, $X \in P$ with $\ell \subseteq \Gamma_k(x), k \geq 3$. We must show $\ell^{\perp} \cap \Gamma_{k-1}(x) \neq \emptyset$ a singular subspace. Let $y \in \ell$, and $z \in y^{\perp} \cap \Gamma_{k-1}(x)$. We assert that $z \geq y \cap x$. If not, then there is linear three-subspace, N, contained in $z \cap x$, with $y \cap N = \emptyset$. Then $z \cap y \subseteq N' \cap y$ (here N' is the collection of all vectors of V orthogonal to N), but $\dim z \cap y = n-2$, $\dim N' \cap y = n-3$, so we

have a contradiction. Thus our assertion follows.

Now set $\mathcal{U} = \bigcap_{y \in \ell} y$, so $\mathcal{U}$ is a totally singular n − 2 subspace of V. Since $\ell \subseteq \Gamma_k(x)$, dim ${}^{x}/x \cap y = 2k$ for each $y \in \ell$. Then we must have dim $\mathcal{U} \cap x$ n − 1 − 2k and dim $\mathcal{U} \cap x = n + 1 - 2k$, so that there is a subspace A of dimension two in $\mathcal{U}' \cap x$ complimenting $\mathcal{U} \cap x$. Set $M = \mathcal{U} \oplus A$, $N = M \cap x$. Note that $M \in \mathcal{M} \setminus P$. Let

$$\Delta = \{z = (M \cap W') + W : W \subseteq x, W \supseteq M \cap x, \dim {}^{W}/M \cap x = 1\} .$$

Then clearly Δ is a singular subspace of (P,L) with rank 2k − 2, and $\Delta \subseteq \ell^{\perp} \cap \Gamma_{k-1}(x)$. Thus to prove the proposition it suffices to prove $\ell^{\perp} \cap \Gamma_{k-1}(x) \subseteq \Delta$.

Let $z \in \ell^{\perp} \cap \Gamma_{k-1}(x)$. Then from the very beginning of the proof $z \supseteq \langle y \cap x : y \in \ell \rangle = \mathcal{U}' \cap x = M \cap x$. Now since dim $z \cap x = n + 2 - 2k$, if $W = z \cap x$, then W contains $M \cap x$ as a hyperplane. Now z must equal $(W' \cap y) + W$, for each $y \in \ell$. But $(W' \cap y) + W = (M \cap W') + W$ and $z \in \Delta$ as desired.

The main result of this paper is

(1.14) <u>THEOREM</u>. *Let* (P,L) *be an incidence structure whose maximal singular subspaces all have finite rank, and satisfies* (D1)-(D3). *Then either* (P,L) *is a thick, non-degenerate polar space of rank* 4 *or for some* $k \geq 5$ *and field* K, (P,L) *is isomorphic to* $D_{n,n}(K)$.

2. PRELIMINARY LEMMAS

(2.1) <u>LEMMA</u>. *Let* $y \in \Gamma_2(x)$. *Then* $S(x,y) = \langle x, y, \{x,y\}^{\perp} \rangle$ *is a polar spce of rank four. Moreover, if* $x', y' \in S(x,y)$ *with* $y' \notin (x')^{\perp}$, *then* $S(x',y') = S(x,y)$.

<u>PROOF</u>. See (3.9) and the corollary to (3.11) in [4].

(2.2) <u>NOTATION</u>. The subspaces $S(x,y) = \langle x, y\}^{\perp} \rangle$, where $d(x,y) = 2$, will be called *Symplectons* or *Symps*. We denote the collection of all symps by <u>Symp</u>.

(2.3) LEMMA. *If* $x \in P$, $\ell \in L$ *with* $\ell \subseteq x^{\perp} \setminus \{x\}$, then there is an $S \in$ <u>Symp</u>, PROOF. See (3.12) of [4].

(2.4) COROLLARY. *If* $M \in \underline{\underline{Sing}}$, *then* $(M, L(M))$ *is a Desarguesion projective space.*

PROOF. By VEBLEN and YOUNG [6], we need only prove the result if $M = \langle \ell, x \rangle$ with $x \in P$, $\ell \in L$, $\ell \subseteq x^{\perp} \setminus \{x\}$. However, this case follows from (2.3) and Tits' classification of polar spaces [5].

(2.5) NOTATION. $\underline{\underline{V}}$ is the subset of $\underline{\underline{Sing}}$ of singular subspaces which contain lines as maximal subspaces. We call elements of $\underline{\underline{V}}$ *planes*.

(2.6) LEMMA. *If there exists a pair* $x, w \in P$ *with* $d(x,w) = 2$ *and for each* $\ell \in L_x$, $\ell \cap \Gamma(w) \neq \emptyset$, *then* (P, L) *is a thick, nondegenerate polar space of rank* 4.

PROOF. See (3.13) of [4].

3. INCIDENCE STRUCTURES INDUCED AT A POINT

In this section we induce an incidence structure at a point, called the residue of the point and identify its structure. Thus, let $x \in P$. The points of the residue are the lines on x, L_x, the lines are the planes on x, $\underline{\underline{V}}_x$, with ordinary inclusion as incidence. Thus, if ℓ, $m \in L_x$, ℓ, m will be collinear in the residue if and only if $m \subseteq \ell^{\perp}$, and then the line on ℓ and m is $L_x(\langle \ell, m \rangle)$. For $\ell \in L_x$, $\Gamma_x(\ell) = \{m \in L_x(\ell^{\perp}) - \{\ell\}\}$. We first prove

(3.1) LEMMA. $(L_x, \underline{\underline{V}}_x)$ *is a thick, gamma space whose point graph* (L_x, Γ_x) *has diameter two and satisfies*

(A1) *It* ℓ, $m \in L_x$ *and* $m \notin \Gamma_x(\ell)$, *then* $\Gamma_x(\ell) \cap \Gamma_x(m)$, *together with its lines, is a non-degenerate generalized quadrangle and*

(A2) *If* $V \in \underline{\underline{V}}_x$, $\ell \in L_x$ *such that* $L_x(V) \cap \Gamma_x(\ell) = \emptyset$, *and* $C_x(V, \ell) = \langle m \in L_x : \ell, L_x(V) \subseteq \Gamma_x(m) \rangle \in \underline{\underline{V}}_x$.

PROOF. Clearly $(L_x, \underline{\underline{V}}_x)$ is thick. We first show (L_x, N_x) is a gamma space. Let $\ell \in L_x, V \in \underline{\underline{V}}_x$ and suppose $|\Gamma_x(\ell) \cap L_x(V)| \geq 2$. Then there are $m_1, m_2 \in L_x, (V)$ such that $m_1, m_2 \subseteq \ell^{\perp}$. Then $V = \langle m_1, m_2 \rangle \subseteq \ell^{\perp}$, and hence

$L_x(V) \subseteq \Gamma_x(\ell)$.

Next suppose $\ell = xa$, $m = xb \in L_x$, $m \notin \Gamma_x(\ell)$. Then $d(a,b) \geq 2$. Since $x \in \{a,b\}^\perp$, $d(a,b) = 2$. Then $\{a,b\}^\perp$ is a polar-space of rank 3, in particular $\{a,b\}^\perp \cap x^\perp \neq \emptyset$. If $c \in \{a,b\}^\perp \cap x^\perp$, then $xc \in \Gamma_x(\ell) \cap \Gamma_x(m)$, so diam $\{L_x,\Gamma_x\} = 2$. Also see that $\Gamma_x(\ell) \cap \Gamma_x(m) = L_x(\{a,b\}^\perp)$, and so is a non-degenerate generalized quadrangle. Therefore (A1) is satisfied.

Finally, suppose $V \in \underline{\underline{V}}_x$, $\ell \in L_x$, $\Gamma_x(\ell) \cap L_x(V) = \emptyset$. Let $k \in L(V)\backslash L_x$, $a \in \ell\backslash\{x\}$. Then $a^\perp \cap m = \emptyset$. However, $a^\perp \cap m^\perp \neq \emptyset$, since $x \in a^\perp \cap m^\perp$. Therefore $a^\perp \cap m^\perp \in \underline{\underline{V}}_x$. It is clear to see that $C_x(V,\ell) = a^\perp \cap m^\perp$, and the lemma is completed.

(3.2) <u>COROLLARY</u>. *For each* x, *there is an integer* $N_x \geq 3$, *and division ring* K_x *such that* $(L_x,\underline{\underline{V}}_x)$ *is isomorphic to* $A_{n_x,2}(K_x)$.

<u>PROOF</u>. Here $A_{n,2}(K)$ is the gamma space whose points are the projective lines in $PG(n+1,K)$, and the lines are in one-one corresponse with incident pairs (π_0,π_2) where $_0$ is a projective point and π_2 a projective plane, and the line is the pencil determined by (π_0,π_2). The corollary follows from (3.1) and Theorem A of [2] and [4].

(3.3) <u>LEMMA</u>. *The graph* (P,Γ_2) *is connected.*

<u>PROOF</u>. Since (P,Γ) is connected it suffices to prove if $y \in \Gamma(x)$, then $\Gamma_2(x) \cap \Gamma_2(y) \neq \emptyset$. By (2.3), if $\ell = xy$, then $\underline{\underline{Symp}}_\ell \neq \emptyset$. If $S \in \underline{\underline{Symp}}_\ell$, then $\Gamma_2(x) \cap \Gamma_2(y) \cap S \neq \emptyset$.

(3.4) <u>LEMMA</u>. *For each* $x \in P$, K_x *is a field. Moreover all the* K_x *are isomorphic.*

<u>PROOF</u>. Let $x \in P$, $S \in \underline{\underline{Symp}}_x$, $L_x(S)$ is a Symp of $(L_x,\underline{\underline{V}}_x)$, and so $L_x(S) \cong A_{3,2}(K_x)$. From Tits' classification of polar spaces (see section 8 of [5]), it follows that K_x is a field and $S \cong D_4(K_x)$. To prove the latter part of the lemma it suffices to prove for $d(x,y) = 2$, then $K_x \cong K_y$. Thus if $d(x,y) = 2$, let $S = S(x,y)$. Then $S \cong D_4(K_x)$ and $S \cong D_4(K_y)$. By (6.13) of [5] it follows that $K_x \cong K_y$.

For the sequel we let K be the underlying field. Note that now all singular subspaces are projective spaces over K. Those of rank t we denote by ${}_tP$.

(3.5) LEMMA. *Let* $x,y \in P$. *Then* $n_x = n_x$.

PROOF. By connectedness of (P,Γ) suffices to prove $n_x = n_y$ for $y \in \Gamma(x)$. Set $\ell = xy$. Then $\ell \in L_x$, and $(L_x,\underline{\underline{V}}_x) = A_{n_x,2}(K)$. Then if $M \in \underline{\underline{Sing}}_\ell$ is choosen so that rk (M) is maximal, then as a singular subspace of $(L_x,\underline{\underline{V}}_x)$, $(L_x(M)) = n_x-1$. It therefore follows that rk (M) $= n_x$. By similarly considering $(L_y,\underline{\underline{V}}_y)$, we see rk (M) $= n_y$ and so $n_x = n_y$ as claimed.

4. PROOF OF THE MAIN THEOREM

We now have that there is an integer $n \geq 3$, and field K such that for each point x in P, $(L_x,\underline{\underline{V}}_x) \cong A_{n,2}(K)$. We will prove by induction on n that $(P,L) \cong D_{n+1,n+1}(K)$.

(4.1) LEMMA. *If* $n = 3$, *then* $(P,L) \cong D_4(K) \cong D_{4,4}(K)$.

PROOF. Let $d(x,w) = 2$, $S = S(x,w)$. Then in section three we saw $S \cong D_4(K)$. However, it follows that $x^\perp \subseteq S$, and so by (2.5) that $P = S$.

(4.2) NOTATION. π_x will denote a projective space of rank n over K which underlies $(L_x,\underline{\underline{V}}_x)$.
$R_t = \{x,X\}$ $x \in X \subseteq x^\perp$, $X \in \underline{\underline{Sub}}$, $L_x(X) \cong A_{t,2}(K)$. For $(x,X) \in R_t$, $y \in X - \{x\}$, we set X_y equal to

$$\bigcup_{z\in X-y^\perp} [S(y,z) \cap y^\perp].$$

Finally let $P^+ = {}_n\mathcal{P}$ and $\mathcal{P}^- = \{M \in {}_3P : M^\perp = M\}$.

(4.3) LEMMA. *Let* $S \in \underline{\underline{Symp}}$, $x \in P\backslash S$. *If* $L(S \cap x^\perp) \neq \emptyset$, *then* $S \cap x^\perp \in {}_3P\backslash P^-$.

PROOF. Clearly $S \cap x^\perp \in \underline{\underline{Sing}}$ by (2.1), let $\ell \in L(S\cap x^\perp)$ and $y \in \ell$. Set $m = x_y$. Consider L_y. There is a subspace $\pi_y(S)$ of π_y of rank three such that $L_y(S)$ consists of all lines of $\pi_y(S)$. Now $\ell \in \Gamma_y(m) \cap L_y(S)$, and, therefore, the line of π_y which m is identified with meets $\pi_y(S)$. Then $\Gamma_y(m) \cap L_y(S)$

is a singular plane of L_y. Now it follows that $S \cap x^{\perp} \in {}_3P$. As $x \notin S \cap x^{\perp}$, $S \cap x^{T} \in {}_3P \backslash P^{-}$.

(4.4) LEMMA. *Assume* $S_1, S_2 \in \underline{\underline{\mathrm{Symp}}}$ *and* $\underline{\underline{V}}(S_1 \cap S_2) \neq \emptyset$. *Then* $S_1 \cup S_2 \in P^{-}$.

PROOF. By (2.1), $S_1 \cap S_2 \in \underline{\underline{\mathrm{Sing}}}$. Let $x \in S_1 \cap S_2$. $L_x(S_i)$ are symps of L_x, and since $\underline{\underline{V}}(S_1 \cap S_2) \neq \emptyset$, $L_x(S_1) \cap L_x(S_2) = L_x(S_1 \cap S_2)$ contains a line of $(L_x, \underline{\underline{V}}_x)$. It then follows that $L_x(S_1 \cap S_2)$ is a maximal singular subspace of rank two, hence, $S_1 \cap S_2 \in P^{-}$.

(4.5) LEMMA. *Let* $(x,X) \in R_t$, $y \in X - \{x\}$. *Then* $(y,X_y) \in R_t$.

PROOF. If $t = 3$, then the result is immediate: for any $z \in X - y^{\perp}$, $X = S(y,z) \cap x^{\perp}$. Then $X_y = S(y,z) \cap y^{\perp}$ and $(y,X_y) \in R_3$, we proceed to prove the lemma in a sequence of short steps. We first introduce some notation. $\underline{\underline{\mathrm{Symp}}}_x(X) = \{S \in \underline{\underline{\mathrm{Symp}}}_x : S \cap x^{\perp} \subseteq X\}$.

I. $X_y \in \underline{\underline{\mathrm{Sub}}}$: Let $u_1, u_2 \in X_y$ with $u_2 \in u_1^{\perp}$. If $u_2 \in yu_1$ then result is clear. Let $S_i \in \underline{\underline{\mathrm{Symp}}}_x(X)$ with $yu_i \subseteq S_i$, $i = 1,2$. If $S_1 = S_2$, then the result is obvious, so we may assume $S_1 \neq S_2$. In particular we may assume $u_1, u_2 \in \Gamma_2(x)$, so $S_i = S(x,u_i)$. Now since $S_1 \cap u_2^{\perp} \supseteq yu_1$, by (4.3), $S_1 \cap u_2^{\perp} \in {}_3P \backslash P^{-}$. Then $S_1 \cap u_2^{\perp} \cap x^{\perp} \in {}_2P$, and hence by (4.4), $\langle x, S_1 \cap u_2^{\perp} \cap x^{\perp} \rangle = S_1 \cap S_2 \in P^{-}$. Set $M = S_1 \cap S_2$.
Note that $u_1^{\perp} \cap M = u_2^{\perp} \cap M$. Let $N \in {}_2P_x(M)$, i.e. a hyperplane of M containing x, with $y \notin N$. Let $\{M_i\} \in {}_3P_N(S_i)$, $i = 1,2, M_i \neq M$ (there are unique such choices). Then by consideration of L_x we see that $M_2 \subseteq M_1^{\perp}$ and $\langle M_1, M_2 \rangle \in {}_4P$. Let $v_i \in M_i \cap u_i^{\perp} \backslash M$, $i = 1,2$. Now $v_1 \notin u_2^{\perp}$, for if $v_1 \in u_2^{\perp}$, then $v_1 \in \{u_2,x\}^{\perp} \cap S_1 \subseteq S_1 \cap S_2 = M$, a contradiction. However, $u_1, u_2, v_1, v_2 \in S(u_2, v_1)$, a symp, and so $u^{\perp} \cap v_1v_2$ is a point, say v. Now $v \notin y^{\perp}$, for if $v \in y^{\perp}$, then $v \in \{v_1,v_2\}^{\perp} \cap y^{\perp} \subseteq S_1 \cap S_2 = M$. But then $v_2 \in \langle M, v_1 \rangle \subseteq S_1$, a contradiction. Thus $S(u,x) = S(y,s)$. Since $v_1, v_2 \in X$ and X is a subspace, $v \in X$. Hence $S(u,x) \in \underline{\underline{\mathrm{Symp}}}_x(X)$ and $u \in X_y$.

II. If $u_1, u_2 \in X_y$, $d(u_1,u_2) = 2$, then $S(u_1,u_2) \cap y^{\perp} \subseteq X_y$.
Pf: Let $S_i \in \underline{\underline{\mathrm{Symp}}}_x(X) \cap \underline{\underline{\mathrm{Symp}}}_{x_i}$, $i = 1,2$. If $S_1 = S_2$, then the result is clear, so assume $S_1 \neq S_2$. Then we may also assume $u_1, u_2 \in \Gamma_2(x)$.
Let $v \in \{u_1,u_2\}^{\perp} \cap y^{\perp}$. If $v \in x^{\perp}$, then $v \in \{x,u\}^{\perp} \subseteq S_1$, so $v \in X_y$ in this

case. Thus assume $v \in \Gamma_2(x)$. Now consider L_y. The three subspace $\pi_y(S_i)$ of π_i meet in a plane $\mathcal{U}$, and this plane contains the line which xy is identified with. The lines which u_iy are identified with meet $\mathcal{U}$ in projective points ρ_i moreover, since $(u_1,u_2),(u_i,x) \in \Gamma_2, \rho_i$ are not on xy and, $\rho_1 \neq \rho_2$. Now vy "meets" both u_1y and u_2y. If ρ_i is on vy for some i, then vy is contained in $\pi_y(S_j)$, where $\{i,j\} = \{1,2\}$, that is $vy \in L_y(S_j)$ and $v \in S_j$, in which case $v \in X_y$. Thus u_iy "meets" vy in a point $\delta_i \neq \rho_i$, $i = 1,2$. From this it follows that there are lines $m_i = w_iy \in L_y(S_i) \cap \Gamma_y(xy) \cap \Gamma_y(vy)$ with $m_1 \in \Gamma_y(m_2)$ (choose lines m_i to contain δ_i and meet xy in points q_i with $q_1 \neq q_2$). Now $w_i \in S_i \cap v^\perp \cap x^\perp \cap y^\perp$ and so $w_i \in X$, also $d(w_1,w_2) = 2$. Since $y,v \in \{w_1,w_2\}^\perp$, $S(w_1,w_2) \in \underline{\underline{\text{Symp}}}_x(X) \cap \underline{\underline{\text{Symp}}}_y$. As $v \in S(w_1,w_2) \cap y^\perp$ it follows that $v \in X_y$.

III. $X_y \cap x^\perp = X \cap y^\perp$

Pf: Let $z \in X \cap y^\perp$. Then clearly $X \cap z^\perp \backslash y^\perp \neq \emptyset$. Let $w \in X \cap z^\perp \backslash y^\perp$. Then $z \in S(y,w) \cap y^\perp \subseteq X_y$. Thus $z \in X_y \cap x^\perp$ and we have shown $X \cap y^\perp \subseteq X_y \cap x^\perp$. Conversely, suppose $z \in X_y \cap x^\perp$. Let $S \in \underline{\underline{\text{Symp}}}_x(X) \cap \underline{\underline{\text{Symp}}}_{yz}$. Then $z \in S \cap x^\perp \cap y^\perp \subseteq X \cap y^\perp$, and we have equality.

IV. $(y,X_y) \in \mathcal{R}_t$.

Pf: From I. and II., $L_y(X_y)$ is a subspace of L_y, is connected, has diameter two, and is 2-closed (i.e. if $m_1,m_2 \in L_y(X_y)$ with $m_1 \notin \Gamma_y(m_2)$, then $\Gamma_y(m_1) \cap \Gamma_y(m_2) \subseteq L_y(X_y)$. From this it follows that $L_y(X_y) \cong A_{t',2}(K)$ for some t'. Now let $M \in {}_t\mathcal{P}_{xy}(X)$. Then $M \subseteq X \cap y^\perp = xy \cap x^\perp$. Hence $M \in {}_t\mathcal{P}_{xy}(X_y)$ and so $t \leq t'$. On the other hand, by choosing $M' \in {}_{t'}\mathcal{P}_{xy}(X_y)$ we get $M' \in {}_{t'}\mathcal{P}_{xy}(X)$, and so $t' \leq t$. Thus $t = t'$ and the lemma is proved.

(4.6) DEFINITION. For (x,X), $(y,Y) \in \mathcal{R}_t$, write $(x,X) \sim (y,Y)$ if $(x,X) = (y,Y)$ or if there exists a sequence $\{(x_i,X_i)\}_{i=0}^s \subseteq \mathcal{R}_t$ with $(x_0,X_0) = (x,X)$, $(x_s,X_s) = (y,Y)$ and such that for each i, $x_{i+1} \in X_i$ and $(X_i)_{x_{i+1}} = X_{i+1}$.

Suppose $(x,X) \in \mathcal{R}_t, y \in \mathcal{P}$, and $\pi = (x_0,x_1,\ldots,x_s)$ a path from x to y. We shall say X_π is *defined* if there exists a sequence $\{(x_i,X_i)\}_{i=0}^s$ in $\mathcal{R}_t$ such that $x_i \in X_i$ and $(X_i)_{x_{i+1}} = X_{i+1}$. When X_π is defined each X_i is uniquely determined and we set $X_\pi = X_s$.

(4.7) <u>LEMMA</u>. *Let* $(x,X) \in \mathcal{R}_t, y \in x^\perp \backslash X$. *Set* $Y = \cup_{z \in X \backslash y^\perp} [S(y,z) \cap y^\perp]$.
(i) *If* $X \cap y^\perp = \{x\}$, *then* $(y,Y) \in \mathcal{R}_{t+2}$.
(ii) *If* $X \cap y^\perp \supsetneq \{x\}$, *then* $(y,Y) \in \mathcal{R}_{t+1}$.

<u>PROOF</u>. In either case $Y = (\bar{X})_y$ where $\bar{X} = \langle X,y \rangle$. In (i) clearly $(x,\bar{X}) \in \mathcal{R}_{t+2}$ and in (ii) $(x,\bar{X}) \in \mathcal{R}_{t+1}$. The result follows from (4.5).

(4.8) <u>LEMMA</u>. *Let* $(x,X) \in \mathcal{R}_t, y \in X - \{x\}$. *Then* $X = (X_y)_x$.

<u>PROOF</u>. Since X, X_y are isomorphic it suffices to prove $X \subseteq (X_y)_x$. Let $u \in X$. If $u \in y^\perp$, then $u \in X_y$. Then since $u \in X_y \cap x^\perp$, $u \in (X_y)_x$. Thus assume $u \in \Gamma_2(y)$. Then $S(u,y) \cap y^\perp \subseteq X_y$, $x \in S(u,y) \cap y^\perp$, but $S(y,y) \cap y^\perp \subseteq x^\perp$. Choose $v \in S(u,y) \cap y^\perp$, $v \in \Gamma_2(x)$. Then $v \in X_y$ and $S(x,v) \cap x^\perp \subseteq (X_y)_x$. But $S(x,v) = S(u,y)$ and hence $u \in S(x,v) \cap x^\perp \subseteq (X_y)_x$.

(4.9) <u>LEMMA</u>. *Let* $(x,X) \in \mathcal{R}_t$, $a,b \in X - \{x\}$ *with* $b \in a^\perp$. *Then* $(X_a)_b = X_b$.

<u>PROOF</u>. Since $(X_a)_b, X_b \in (\mathcal{R}_t)_b$, it suffices to prove $X_b \subseteq (X_a)_b$. Let $d \in X - b^\perp$, $c \in S(b,d) \cap b^\perp$. Suppose first that $d \in a^\perp$. Then $d \in X_a$ and then $S(b,d) \cap b^\perp \subseteq (X_a)_b$. Thus we may assume $d \in \Gamma_2(a)$.

Since $(X_a)_b$ is a subspace it suffices to show $bc \cap (X_a)_b \neq \{b\}$. Since $b \in \Gamma_2(d)$ and $d^\perp \cap bc \neq \emptyset$, we may assume $c \in d^\perp$. Suppose $cd \cap a^\perp \neq \emptyset$. If $a \in c^\perp$, then $c \in X_a$ and then $c \in X_a \cap b^\perp \subseteq (X_a)_b$. Thus we may assume $c \in \Gamma_2(a)$. Let $c' = cd \cap a^\perp$. Then $c' \in S(a,d) \cap a^\perp \subseteq X_a$ and $c' \in \Gamma_2(b)$. Then $S(b,c') \cap b^\perp \subseteq (X_a)_b$ and this implies $c \in (X_a)_b$. Thus we may assume $cd \subseteq \Gamma_2(a)$.

Suppose now that $x \in c^\perp$. Then $x \in (cd)^\perp \cap a^\perp$. Therefore $a^\perp \cap (cd)^\perp \in {}_2\mathcal{P}(\{a,c\}^\perp)$, and so $a^\perp \cap (cd)^\perp$ is maximal in $\{a,c\}^\perp$. Therefore there is an $e \in a^\perp \cap (cd)^\perp \cap \Gamma_2(b)$. Note $e \in x^\perp$ since $a^\perp \cap (cd)^\perp$ contains x. Since $e \in X \cap a^\perp$, $e \in X_a$. Thus $S(b,e) \cap b^\perp \subseteq (X_a)_b$. However, $c \in b^\perp \cap e^\perp$, so $c \in (X_a)_b$.

Therefore we may assume $x \notin (cd)^\perp$. In particular $c \in \Gamma_2(x)$. Now note that $S(b,d) \supseteq dc$ and $S(b,d) \cap a^\perp \supseteq bx$. Then by (4.3) $S(b,d) \cap a^\perp \in {}_3\mathcal{P}$. If $M = S(b,d) \cap a^\perp$, then $M \cap (cd)^\perp \neq \emptyset$, and hence $(cd)^\perp \cap a^\perp \neq \emptyset$, and hence by (D_2), $a^\perp \cap (cd)^\perp \in {}_2\mathcal{P}$. Set $a^\perp \cap (cd)^\perp = N$. $x,b \notin N$. However, N is maximal in $\{a,c\}^\perp$ and $b \in \{a,c\}^\perp \backslash N$. Therefore, there is an $e \in N \backslash b^\perp$. Now $e \in S(a,d) \cap a^\perp$, so $e \in X_a$. $c \in S(b,e) \cap b^\perp$, so $c \in (X_a)_b$ and we have shown $X_b \subseteq (X_a)_b$.

(4.10) LEMMA. (i) *Suppose* $d(x,y) = k \geq 1$. *Then* $y^{\perp} \cap \Gamma_{k-1}(x)$, *together with its lines is isomorphic to* $A_{2k-1,2}(K)$.
(ii) *If* $\ell \subseteq \Gamma_k(x)$, *then* $\ell^{\perp} \cap \Gamma_{k-1}(x) \in {}_{2k-2}\mathcal{P}$.

PROOF. We first show that (ii) is a consequence of (i). Choose $y \in \ell$. By (i) $y^{\perp} \cap \Gamma_{k-1}(x) \cong A_{2k-1,2}(K)$. Set $Y = \langle y^{\perp} \cap \Gamma_{k-1}(x), y\rangle$ so $(y,Y) \in \mathcal{R}_{2k-1}$ and consider L_y, $L_y(Y)$ and ℓ. Now either $\Gamma_y(\ell) \cap L_y(Y) = \emptyset$ or $\Gamma_y(\ell) \cap L_y(Y)$ is a maximal singular subspace of $L_y(Y)$. Thus, either $\ell^{\perp} \cap \Gamma_{k-1}(x) = \emptyset$ or $\ell^{\perp} \cap \Gamma_{k-1}(x) \in {}_{2k-2}\mathcal{P}$. Since $\ell^{\perp} \cap \Gamma_{k-1}(x) \neq \emptyset$ by (D2) and (D3), (ii) now follows.

We prove (i) by induction on $k \geq 1$. (i) is obvious for $k = 1$ and 2. Thus assume (i) is true for $k = t \geq 2$ and suppose $k = t + 1$. Now let $a \in y^{\perp} \cap \Gamma_t(x)$. By induction $a^{\perp} \cap \Gamma_{t-1}(x) \cong A_{2t-1,2}(K)$. Set $A = \langle a, a^{\perp} \cap \Gamma_{t-1}(x)\rangle$, so $(a,A) \in \mathcal{R}_{2t-1}$. Note that for $\ell \in L_a(A)$, $\ell \cap \Gamma_{t-1}(x)$ is a point. Since $y \in \Gamma_{t+1}(x)$, $A \cap y^{\perp} = \{a\}$. Now let $b \in A - \{a\}$, so $b \in \Gamma_2(y)$. Let $c \in S(b,y) \cap y^{\perp}$. Then $yc \cap b^{\perp} \neq \emptyset$, and if $c' \in yc \cap b^{\perp}$, then $c' \in \Gamma_t(x) \cap y^{\perp}$. Thus, if $\ell \in L_y(S(b,y))$, then ℓ contains a unique point in $y^{\perp} \cap \Gamma_t(x)$. Now by (4.7), if $Y = \cup[S(b,y) \cap y^{\perp}]$, $b \in A - \{a\}$ then $(y,Y) \in \mathcal{R}_{2t+1}$. Since each $\ell \in L_y(Y)$ contains a unique point in $y^{\perp} \cap \Gamma_t(x)$, if we set $Z = Y \cap \Gamma_t(x)$, then $Z \cong A_{2t+1,2}(K)$.

We next show that $W = \Gamma_t(x) \cap y^{\perp}$ is a subspace. Suppose $u,v \in \ell \cap W$. Then either $\ell \subseteq W$ or there is a unique point $w \in \ell \cap \Gamma_{t-1}(x)$. But then $d(x,y) \leq d(x,w) + d(w,y) = t - 1 + 1 = t$, a contradiction. Now suppose $a,b \in W$, $d(a,b) = 2$, $c \in \{a,b\}^{\perp} \cap y^{\perp}$. Claim $yc \cap W \neq \emptyset$. If $yc \cap W = \emptyset$, then $yc \subseteq \Gamma_{t+1}(x)$. Then by (D3), $(yc)^{\perp} \cap \Gamma_t(x) \in$ Sing. However, $a,b \in (yc)^{\perp} \cap \Gamma_t(x)$ and $b \notin a^{\perp}$, a contradiction. Thus $yc \cap W \neq \emptyset$. It follows that $\{a,b\}^{\perp} \cap W$ is a non-degenerate generalized quadrangle, and therefore that $W \cong A_{s,2}(K)$ for some $s \geq 2t+1$ (since $W \supseteq Z$). Thus to complete the proof it suffices to prove $s = 2t+1$.

Now let $a \in W$ and $m \in L_a(W)$. Then $m \subseteq \Gamma_t(x)$ and therefore by induction $m^{\perp} \cap \Gamma_{t-1}(x) \in {}_{2t-2}\mathcal{P}(U)$, $U = \Gamma_{t-1}(x) \cap a^{\perp}$. Suppose that $m_1, \in L_a(W)$, but $m_1 \subseteq m_2^{\perp}$. Then $m_1^{\perp} \cap \Gamma_{t-1}(x) \neq m_2^{\perp} \cap \Gamma_{t-1}(x)$. For suppose on the contrary, $m_1^{\perp} \cap \Gamma_{t-1}(x) = m_2^{\perp} \cap \Gamma_{t-1}(x) = M$. Let $b_i \in m_i$, $i = 1,2$. Then $y, M \subseteq b_1^{\perp} \cap b_2^{\perp}$. Then $\emptyset \neq y^{\perp} \cap M \subseteq \Gamma_{t-1}(x) \cap y^{\perp} = \emptyset$, a contradiction.

Next suppose $m_1, m_2 \in L_y(W)$ and $m_1^{\perp} \cap \Gamma_{t-1}(x) = m_2^{\perp} \cap \Gamma_{t-1}(x) = M$. Then by the previous paragraph $m_2 \subseteq m_1^\perp$. Set $N = \langle m_1, m_2\rangle$. Claim $N^\perp \cap W = N$. Since $W \cong A_{s,2}(K)$ if $V \in \underline{\underline{V}}(W)$, then $V^\perp \cap W \in$ <u>Sing</u> and either $V^\perp \cap W \in {}_{s-1}P$ or $V^\perp \cap W = V$. Suppose $N^\perp \cap W \in {}_{s-1}P$. Since $N \in \underline{\underline{V}}$, N lies in two maximal singular subspaces, one of rank 3 and one of rank n. Since $M \subseteq \Gamma_2(y)$, $y \notin \langle M,N\rangle^\perp$. Since $\mathrm{rk}(\langle M,N\rangle) \geq 4$, it follows that $\mathrm{rk}(\langle M,N\rangle^\perp) = n$. $\langle y,N\rangle$ is a singular subspace of rank three on N and $\langle y,N\rangle \cap \langle M,N\rangle^\perp = N$. Therefore $\langle y,N\rangle^\perp = \langle y,N\rangle$. However, we are assuming $\mathrm{rk}(N^\perp \cap W) = s-1$. $N^\perp \cap W \subseteq y^\perp$. Then $\langle y, N^\perp \cap W\rangle$ is a singular subspace, $\langle y, N^\perp \cap W\rangle \supseteq \langle y,N\rangle$ and $\mathrm{rk}(\langle y, N^\perp \cap W\rangle) = s \geq 2t+1 \geq 5$, a contradiction.

Thus, if $m_1^\perp \cap \Gamma_{t-1}(x) = m_2^\perp \cap \Gamma_{t-1}(x)$, then $\langle m_1,m_2\rangle^\perp \cap W = \langle m_1,m_2\rangle$. Suppose now $m_1, m_2 \in L_y(W)$, $m_2 \subseteq m_1^\perp$ and $\langle m_1,m_2\rangle^\perp \cap W = \langle m_1,m_2\rangle$. Set $M_i = m_i^\perp \cap \Gamma_{t-1}(x)$. We prove $N_1 = N_2$. Let $n \in L(N_1)$. Then $n \subseteq \Gamma_2(y)$, but $m_1^\perp \subseteq n^\perp \cap y^\perp$. However, $\mathrm{rk}(n^\perp \cap y^\perp) = 2$ and $n^\perp \cap y^\perp \subseteq W$. Also, from the type of L_a we see that $\langle y, n^\perp \cap y^\perp\rangle \in P^-$. Therefore $n^\perp \cap y^\perp$ is a maximal singular plane of W. However, each line of W lie in a unique singular plane of W which is maximal in W. Since $m_1 \subseteq \langle m_1,m_2\rangle$ and $\langle m_1,m_2\rangle^\perp \cap W = \langle m_1,m_2\rangle$, it follows that $\langle m_1,m_2\rangle = n^\perp \cap y^\perp$. Now $N_1 = m_1^\perp \cap n^\perp \supseteq m_2$. Therefore $N_1 \subseteq N_2$. Since $\mathrm{rk}(N_1) = \mathrm{rk}(N_2)$, $N_1 = N_2$ as claimed.

Now we have shown there is an injective map Φ from $V_{max}(W)$ $\{V \in \underline{\underline{V}}(W) : V^\perp \cap W = W\}$ into ${}_{2t-2}P(U)$. Now for $V_1, V_2 \in V_{max}(W)$, define $\Delta(V_1,V_2) = \{W \in V_{max}(W) : V_i \cap W \in L_a,\ i = 1,2\}$. Set $\lambda(V_1,V_2) = \{V \in V_{max}(W) : V \cap V' \in L_a$, for every $V' \in \Delta(V_1,V_2)\}$. If we set $\Lambda = \{\lambda(v_1,v_2) : V_1 \neq V_2 \in V_{max}(W)\}$, then $(V_{max}(W), \Lambda) \cong PG(s-2,K)$. Now ${}_{2t-2}P(U)$ is naturally isomorphic to $PG(2t-1,K)$. We finally show that Φ is a morphism of projective spaces. Since Φ is injective this will imply $s - 2 \leq 2t - 1$ from which we deduce $s \leq 2t + 1$ as desired.

Let $\lambda = \lambda(v_1 v_2) \in \Lambda$. Set $M_i = v_i^\perp \cap \Gamma_{t-1}(x)$. Then $M_1 \cap M_2 = \{u\}$ is a point. Then $\{y,u\}^\perp \subseteq W$, $\{y,u\}^\perp \cong A_{3,2}(K)$ and $a \in \{y,u\}^\perp$. It is clear to see that $V_{max}(W) \cap \underline{\underline{V}}_a(\{y,u\}^\perp) = \lambda(v_1,v_2)$ and from this our claim now follows.

(4.11) <u>NOTATION</u>. If $d(x,y) = k \geq 2$, set $R(x,y) = \langle x^\perp \cap \Gamma_{k-1}(y), x\rangle$. (So $(x, R(x,y)) \in R_{2k-1}$).

(4.12) <u>LEMMA</u>. *Let* $d(x,y) = k \geq 2$ *and be a geodesic from* x *to* y. *If* $X = R(x,y)$, *then* X_π *is defined. Moreover,* $X_\pi = R(y,x) = \langle y, y^\perp \cap \Gamma_{k-1}(x)\rangle$.

PROOF. Induction on $k \geq 2$. Suppose $k = 2$. Then $X = R(x,y) = x^{\perp} \cap S(x,y) = \langle x,\{x,y\}^{\perp}\rangle$. For $z \in \{x,y\}^{\perp}$, $X_z = z^{\perp} \cap S(x,y)$ and $y \in X_z$. Thus, if $\pi = (x,z,y)$, then X_π is defined and $X_\pi = (X_z)_y = S(x,y) \cap y^{\perp} = \langle y,\{x,y\}^{\perp}\rangle = R(y,x)$.

Assume now that the result is true for all $k \leq t$ and let $k = t + 1$. Let $\pi = (x = x_0, x_1, \ldots, x_{t+1} = y)$ be a geodesic path from x to y. Set $x_1 = a$. We show that $A = R(a,y) = \langle a, \Gamma_{t-1}(y) \cap a^{\perp}\rangle \subseteq X_a$. Of course it suffices to show $\Gamma_{t-1}(y) \cap a^{\perp} \subseteq X_a$ since $X_a \in \underline{\underline{\mathrm{Sub}}}_a$. Let $b \in \Gamma_{t-1}(x) \cap a^{\perp}$, $c \in \{x,b\}^{\perp}$. Then $d(c,y) = t$ and $c \in \Gamma_t(y) \cap x^{\perp}$. Choose $c \in \Gamma_2(a)$. $c \in X = R(x,y)$ and $a^{\perp} \cap S(a,c) \subseteq X_a$. However, $S(a,c) = S(x,b)$ and hence $b \in X_a$. Now if $\rho = (a = x_1, x_2, \ldots, x_t = y)$, then by induction A_ρ is defined. Since $A \subseteq X_a$, $(x_a)\rho$ is defined. But $(X_a)\rho = X_\pi$ and hence X_π is defined. Note by induction we also have $X_\pi \supseteq y^{\perp} \cap \Gamma_{t-1}(a)$. However,

$$\bigcup_{a \in x^{\perp} \cap \Gamma_t(y)} [y^{\perp} \cap \Gamma_{t-1}(a)] = y^{\perp} \cap \Gamma_t(x).$$

Therefore, $X_\pi \supseteq \langle y, y^{\perp} \cap \Gamma_t(x)\rangle = R(y,x)$. Since both (y,X_π) and $(y,R(y,x)) \in \mathcal{R}_{2t+1}$ we have $X_\pi = R(y,x)$.

Now let $(x,X) \in \mathcal{R}_t$. suppose $d(x,y) = k > [\frac{t+1}{2}]$. Then $x^{\perp} \cap \Gamma_{k-1}(y) \cong A_{2k-1,2}$. Since $2k-1 > t$, $x^{\perp} \cap \Gamma_{t-1}(y) \not\subseteq X$. We remark that at this point it now follows diam $(P,\Gamma) = [\frac{n+1}{2}]$.

Now set

$$D(x,X) = \bigcup_{k \geq 1} \{y : d(x,y) = k,\ R(x,y) \subseteq X\} \cup \{x\}$$

(4.12) REMARK. $x^{\perp} \cap D(x,X) = X$

(4.13) LEMMA. *Let* $(x,X) \in \mathcal{R}_t$, $y \in X - \{x\}$, $Y = X_y$. Then $D(x,X) = D(y,Y)$.

PROOF. As $Y_x = X$ by (4.8) it suffices to prove $D(y,Y) \subseteq D(x,X)$. Recall

$$X_y = \bigcup_{z \in x - y^{\perp}} [S(y,z) \cap y^{\perp}].$$

Now let $z \in D(y,Y)$ with $d(y,z) = k$. Of course if $z = x$, then $z \in D(x,X)$. This leaves four cases to consider:

(i) $d(x,z) = k-1 \geq 1$;

(ii) $d(x,z) = k+1$;

(iii) $d(x,z) = k$, $d(xy,z) = k-1$;

(iv) $xy \subseteq \Gamma_k(z)$.

(i) Let $u \in x^\perp \cap \Gamma_{k-2}(x)$. Then $u \in \Gamma_2(y)$. If $v \in \{u,y\}^\perp$, then $v \in \Gamma_{k-1}(z) \cap y^\perp$. Thus $\{u,y\}^\perp \subseteq Y$ and hence $y^\perp \cap S(u,y) = \langle y,\{u,y\}^\perp \rangle \subseteq Y$. Now choose $v \in \{u,y\}^\perp \cap \Gamma_2(x)$. Then $S(y,u) = S(x,v)$. Then $x^\perp \cap S(x,v) = x^\perp \cap S(y,u) \subseteq Y_x = X$. Thus $u \in X$.

(ii) Let $u \in \Gamma_k(z) \cap x^\perp$. Suppose $u \in y^\perp$. Then $d(yu,z) = k$. Let $v \in \Gamma_{k-1}(z) \cap (yu)^\perp$. Then $v \in Y \cap \Gamma_2(x)$. $X = Y_x \supseteq x^\perp \cap S(x,v)$ and so $u \in X$. Thus assume $u \in \Gamma_2(y)$. Now let $v \in \{x,u,y\}^\perp$. $d(z,v') \leq k+1$ for each $v' \in xv$ since $v' \in y^\perp$ and $d(y,z) = k$. However, if $d(xv,z) = k+1$, then $(xv)^\perp \cap \Gamma_k(z) \in$ <u>Sing</u>, contradicting $u,y \in (xv)^\perp \cap \Gamma_k(z)$. Then without loss of generality we may assume $v \in \Gamma_k(z)$. By the first part of this paragraph $v \in X$. Now Rad $(\{x,y,u\}^\perp) = \{x\}$, hence there is a $w \in \{x,y,u\}^\perp \cap \Gamma_2(z)$. Then also $w \in X$. Then $X \supseteq S(v,w) \cap x^\perp$ and so $u \in X$.

(iii) Let $w = xy \cap \Gamma_{k-1}(z)$. Let $u \in \Gamma_{k-1}(z) \cap x^\perp$. If $u \in y^\perp$, then $u \in Y \cap x^\perp \subseteq Y_x = X$. So assume $u \in \Gamma_2(y)$. As in (ii) we can find a,b with $a \in \Gamma_2(b)$, $a,b \in \{x,u,w\}^\perp \cap \Gamma_{k-1}(z)$. Then also $a,b \in y^\perp$ and so $a,b \in \Gamma_{k-1}(z) \cap y^\perp \subseteq Y$. Then $S(a,b) \cap y^\perp \subseteq Y$. As $x \in S(a,b)$ it follows that $S(a,b) \cap x^\perp \subseteq Y_x = X$. Since $u \in a^\perp \cap b^\perp \cap x^\perp$, $u \in X$.

(iv) Let $u \in \Gamma_{k-1}(z) \cap x^\perp$. If $u \in (xy)^\perp$, then $u \in Y \cap x^\perp \subseteq x$. Thus assume $u \in \Gamma_2(y)$. Now $(xy)^\perp \cap \Gamma_{k-1}(Z) \in {}_{2k-2}\mathcal{P}$ and $u \notin (xy)^\perp \cap \Gamma_{k-1}(Z)$. Clearly, we may assume $k > 1$, for otherwise $u = x$. Thus $u^\perp \cap (xy)^\perp \cap \Gamma_{k-1}(z) \in L$. Then we can find $v \in \Gamma_2(u) \cap (xy)^\perp \cap \Gamma_{k-1}(z)$. Let $a \in \{x,u,v\}^\perp$. Since $u \in (a')^\perp \cap \Gamma_{k-1}(z)$ for each $a' \in ax$, $d(z,a') \leq k$. However, if $d(xa,z) = k$ we get a contradiction : $u,v \in (ax)^\perp \cap \Gamma_{k-1}(z) \in$ <u>Sing</u>. Therefore $d(xa,z) = k-1$, so without loss we may assume $a \in \Gamma_{k-1}(z)$ and $av \subseteq \Gamma_{k-1}(z)$. Let $b \in \Gamma_{k-2}(z) \cap (av)^\perp$. Since $v \in \{y,b\}^\perp$, $d(y,b) = 2$. Since $\{y,b\}^\perp \subseteq \Gamma_{k-1}(z)$, $S(y,b) \cap y^\perp \subseteq Y$. Consequently, $Y_v \supseteq S(y,b) \cap v^\perp$. Since $v \in Y \cap x^\perp$, $v \in X$. Since $b \in S(y,b) \cap v^\perp$, $b \in Y_v$. As $x \in Y \cap v^\perp$, $x \in Y_v$.

Now $d(x,b) = 2$, so $x^{\perp} \cap S(x,b) \subseteq (Y_v)_x = Y_x = X$ by (4.9). As $a \in \{x,b\}^{\perp}$, $a \in X$. However, Rad $(\{x,u,v\}^{\perp}) = \{x\}$, so we can find a $c \in \{x,u,v\}^{\perp} \cap \Gamma_{k-1}(z)$ with $c \in \Gamma_2(a)$. Then as above, $c \in X$. Then $x^{\perp} \cap S(a,c) \subseteq X$, and so $u \in \{a,c,x\}^{\perp} \subseteq S(a,c) \cap x^{\perp}$.

(4.14) COROLLARY. *Let* $(x,X) \in \mathcal{R}_t$, $y \in D(x,X)$ *and* π *a geodesic from* x *to* y, *then* X_π *is defined*, $X_\pi = D(x,X) \cap y^{\perp}$ *and if* $Y = X_\pi$, *then* $D(x,X) = D(y,Y)$

PROOF. This follows from (4.13) and induction on $d(x,y)$.

(4.15) REMARK. The corollary implies that $D(x,X) \in \underline{\underline{\text{Sub}}}$ and for any $a,b \in D(x,X)$ and every geodesic path π from a to b is contained in $D(x,X)$. It follows that $D(x,X)$ satisfies the hypotheses of the main theorem. Thus, if $t < n$, then by induction $D(x,X) \cong D_{t+1,t+1}(K)$.

Now set $\overline{P_{t+1}} = \{D(x,X) : (x,X) \in \mathcal{R}_t\}$, $\overline{P} = \overline{P_{n-1}}$. For $D_1,D_2 \in \overline{P}$, define $D_1 \approx D_2$ if and only if $D_1 \cap D_2 \neq \emptyset$.

Now suppose $D_1,D_2 \in \overline{P}$, $D_1 \approx D_2$. Let $x \in D_1 \cap D_2$. By considering L_x, $L_x(D_i)$, $i = 1,2$, we see that $L_x(D_1 \cap D_2) = L_x(D_1) \cap L_x(D_2) \cong A_{n-1,2}$. Since this is true for each $x \in D_1 \cap D_2$ we have

(4.16) LEMMA. *If* $D_1,D_2 \in \overline{P}$, $D_1 \neq D_2$ *and* $D_1 \cap D_2 \neq \emptyset$, *then* $D_1 \cap D_2 \in \overline{P_{n-2}}$.

Now if $D_1 \approx D_2$, set $\ell(D_1,D_2) = \{D \in \overline{P} : D \supseteq D_1 \cap D_2\}$ and $\overline{L} = \{\ell(D_1,D_2) : D_1,D_2 \in \overline{P}, D_1 \approx D_2\}$. Thus we have an incidence structure $(\overline{P},\overline{L})$.

(4.17) LEMMA. *Let* $D \in \overline{P}$, $x \in P - D$. *If* $\Gamma_2(x) \cap D \neq \emptyset$, *then* $x^{\perp} \cap D \neq \emptyset$.

PROOF. Let $w \in \Gamma_2(x) \cap D$. $L_w(D) \cong A_{n-1,2}(K)$, $L_w(S(x,w)) \cong A_{3,2}$, let $\pi_w(D)$ be the hyperplane of π_w underlying $L_w(D)$ and $\pi_w(x)$ the three subspace underlying $L_w(S(x,w))$. Then $\pi_w(x)$ meets $\pi_w(D)$ in a least a plane so $L_w(D) \cap L_w(S(x,w))$ a contains a singular plane of L_w. Therefore ${}_3\mathcal{P}(S(x,w) \cap D) \neq \emptyset$. If $M \in {}_3\mathcal{P}(S(x,w) \cap D)$, then $M \cap x^{\perp} \in \underline{\underline{V}}(D)$, in particular $D \cap x^{\perp} \neq \emptyset$ as claimed.

(4.18) LEMMA. *If* $D \in \overline{P}$, $x \in P - D$, *then* $x^{\perp} \cap D \neq \emptyset$.

PROOF. Set $s = d(D,x)$. Wish to prove $s = 1$. Suppose on the contrary that $s > 1$. Choose $z \in D$ with $d(x,z) = s$ and let $x = x_0, x_1, \ldots, x_s = z$ be a geodesic from x to z. Let $y = x_{s-2}$. Then $d(x,y) = s-2$.
Since $d(s,x) = s$, $y \in P - D$. Since $\Gamma_2(y) \cap D \neq \emptyset$, by (4.17) $y^\perp \cap D \neq \emptyset$.
If $w \in y^\perp \cap D$, then $w \in D$ and $d(x,w) \leq s-1$, a contradiction. Therefore $s = 1$.

(4.19) NOTATION. For $x \in P$, $\hat{x} = \{D \in \overline{P} : x \in D\}$. For $D \in \overline{P}$, $\Delta(D) = \{D' : D \approx D'\}$.

(4.20) LEMMA. $\hat{x}$, *together with its lines, is a projective space of rank* n *over* K.

PROOF. Clearly $\hat{x}$ is a singular subspace of $(\overline{P},\overline{L})$. We define a map from $\hat{x}$ to $\{X : (X : (x,X) \in R_{n-1}\}$ by $D \mapsto D \cap x^\perp$. Suppose $D_1, D_2 \in \hat{x}$. Then this map carries $\lambda(D_1,D_2)$ to $\{X : (x,X) \in R_{n-1}, X \supseteq D_1 \cap D_2 \cap x^\perp\}$. However, $(x, D_1 \cap D_2 \cap x^\perp) \in R_{n-2}$. Then $\hat{x}$, together with its lines is isomorphic to the incidence structure whose points are the hyperplanes of Π_x ($\cong PG(n,K)$) and lines are the subspaces of codimension two with inclusion as incidence. This is of course a projective space of rank n over K as claimed.

(4.21) LEMMA. *Suppose* $x \notin D \in \overline{P}$. *Then* $\hat{x} \cap \Delta(D)$ *is a hyperplane of* $\hat{x}$.

PROOF. We know $D \cap x^\perp \neq \emptyset$. Since D is geodisically closed, $x^\perp \cap D \in$ Sing. Let $y \in D \cap x^\perp$, $\pi_y(D)$ the hyperplane of π_y underlying $L_y(D)$. The line which xy is identified with meets $\pi_y(D)$. Then $\Gamma_y(xy) \cap L_y(D)$ is a singular subspace of L_y of rank n - 2 and therefore $rk(D \cap x^\perp \cap y^\perp) = n-1$. Since $y \in D \cap x^\perp \in$ Sing, $D \cap x^\perp = D \cap x^\perp \cap y^\perp$. Set $N = D \cap x^\perp$. $rk(\langle N,x \rangle) = n$, and so $M = \langle N,x \rangle \in P^+ = {}_nP$. Then $L_x(M)$ is a maximal singular subspace of rank n - 1 and consists of all lines of Π_x lying on a point Π_D of Π_x. Now suppose $D' \in \hat{x}$ and $D \cap D' \neq \emptyset$. Then $D \cap D' \in \overline{P}_{n-2}$ and $x \in D' - (D \cap D')$. By the above $K = D \cap x^\perp \in {}_{n-2}P$ and $rk(\langle P \cap D' \cap x^\perp, x \rangle) = n-1$. Set $K = \langle D \cap D' \cap x^\perp, x \rangle$, $L_x(K)$ is a singular subspace of L_x of rank n - 2. If $\Pi_x(D')$ is the hyperplane of Π_x corresponding to $L_x(D')$, then $\Pi_x(D')$ contain Π_D. It now follows that $\Delta(D) \cap \hat{x} = \{D' \in \hat{x} : \Pi_k(D') \supseteq \Pi_D\}$ and this is a hyperplane of $\hat{x}$.

The next two results finish the proof.

(4.22) PROPOSITION. $(\overline{P},\overline{L})$ *is a thick, non-degenerate polar space,* $D_{n+1}(K)$.

PROOF. Clearly $(\overline{P},\overline{L})$ is thick. Let $\lambda = \lambda(D_1,D_2) \in \overline{L}$, $D \in \overline{P}$. Let $x \in D_1 \cap D_2$. If $x \in D$, then $\lambda \subseteq \Delta(D)$, so assume $x \notin D$. Then $\Delta(D) \cap \hat{x}$ is a hyperplane of $\hat{x}$ by (4.2), in particular either $\lambda \subseteq \Delta(D)$ or $|\lambda \cap \Delta(D)| = 1$. Thus $(\overline{P},\overline{L})$ is a polar space. Now suppose $D \in \overline{P}$. If $y \in D$, then $L_y(D) \cong A_{n-1,2}(K)$. Since $L_y \cong A_{n,2}(K)$, $y^\perp \not\subseteq D$, so $D \neq P$. If $x \in P - D$, then by (4.21) $\hat{x} \not\subseteq \Delta(D)$, so $D \notin \mathrm{Rad}(\overline{P})$ and as D was arbitrary, $\mathrm{Rad}(\overline{P}) = \emptyset$. Also by (4.21), $\hat{x}$ is a maximal singular subspace of $(\overline{P},\overline{L})$ and so by (4.20), $\mathrm{rk}(\overline{P},\overline{L}) = n+1$.
To see that this is of type D it suffices to show that the residue at a point D of $\overline{P}$, $\overline{L}_D$, is of type D. The map

$$\lambda \longmapsto \bigcap_{D'\in\lambda} D' \quad \text{from } \overline{L}_D \text{ to } \overline{P}_{n-1}(D) \text{ is a bijective morphism}$$

(lines of $\overline{L}_D$ go to $\overline{P}_{n-2}(D)$, and the latter is a polar space $D_n(K)$. This completes the proposition.

THEOREM. $(P,L) \cong D_{n+1,n+1}(K)$

PROOF. The map $x \longmapsto \hat{x}$ is a map from P onto a subset of the maximal singular subspaces of $(\overline{P},\overline{L})$. Now if $\ell \in L_x$, then $\ell = \bigcap_{y\in\ell} \hat{y}$ is easily seen to have rank $k-1$ by passing to $L_x(D \in \hat{\ell}$ if and only if the hyperplane $\pi_x(D)$ contains the line "xy" of π_x). From this it follows that $\{\hat{x} : x \in P\}$ is contained in a single class and $y \in x^\perp$ if and only if $\mathrm{rk}(\hat{x} \cap \hat{y}) = \mathrm{rk}(\hat{x}) - 2 = \mathrm{rk}(\hat{y}) - 2$. Since $L_x \cong A_{n,2}(K)$ it follows that $\{\hat{x} : x \in P\}$ is an entire class and the proof is complete.

5. NEAR 2n-Gons

In this section we recall the definition of a near 2n-gons as introduced by SHULT and YANUSHKA [8], and some related notions.

(5.1) DEFINITION. An incidence structure (P,L) with point-graph (P,Δ) and metric $d(\ ,\ ) = d_\Delta(\ ,\ )$ is a near 2n-gon if (P,Δ) is connected with diameter n and for any pair $(x,\ell) \in P \times L$ with $d(x,\ell) = t$, there is a unique $y \in \ell$ with $d(x,y) = t$.

$y \in \ell$ with $d(x,y) = t$.

(5.2) REMARK. If (P,Δ) is a bipartite graph, then (P,Δ) is a near 2n-gon for some n. In this case lines all have two points. Conversely, a near 2n-gon with two points on each line is bipartite graph. We will refer to such near-2n-gons as thin.

(5.3) NOTATION. For $x \in P$, $\Delta(x)$ is as usual and $x^\perp = \Delta(x) \cup \{x\}$.

(5.4) DEFINITION. A subset X of P is 2-closed if, whenever $x, y \in X, d(x,y) = 2$, then $x^\perp \cap y^\perp \subseteq X$.

(5.5) DEFINITION. In a near 2n-gon, a quad is a subset Q of P satisfying

(i) Q is 2-closed

(ii) diam $(Q,\Delta|Q) = 2$

(iii) Q contains an ordinary quadrangle

Note a quad, together with its lines is a generalized quadrangle.

(5.6) DEFINITION. (i) In a near 2n-gon (P,L) we say quads exists if whenever $d(x,y) = 2$ there exists a quad containing x and y.

(ii) Let $x \in P$, Q a quad of (P,L). The pair (x,Q) is classical if there is a unique point $y \in Q$ with $d(x,Q) = d(x,y) = d$ and $\{z \in Q : d(x,z) = d+1\} = Q \cap y^\perp$.

(5.7) DEFINITION. A dual polar space is the incidence structure whose points are the maximal isotropic (singular) subspaces of a non-degenerate polar space and whose lines are the next to maximal isotropic subspaces.

Note when the polar space is of type D_n the near 2n-gon is thin.

Cumeron has the following characterization of dual polar spaces [9].

(5.8) THEOREM. *An incidence structure* (P,L) *is a dual polar space of rank* n *if and only if the following hold*

(i) (P,L) *is a near* 2n-gon;

(ii) *quads exist;*

(iii) *every point-quad pair is classical.*

We give a proof of this in the case that (P,L) is thin using our main theorem. More precisely we prove.

(5.9) THEOREM. *Let* (P,Δ) *be a connected bipartite graph of diameter* $n \geq 3$. *Further assume*

(i) *If* $d(x,y) = 2$, *then* $|x^{\perp} \cap y^{\perp}| > 2$;

(ii) *In the near* 2n-gon (P,Δ) *quads exist and all point-quad pairs are classical.*

Then one of the following occurs

(i) $n = 3$, *there is a* skew *field* K *such that* (P,Δ) *is the dual polar space of type* $D_3(K)$; *or*

(ii) $n \geq 4$, *there is a field* K *such that* (P,Δ) *is the dual polar space of type* $D_n(K)$.

6. CHARACTERIZATION OF THIN CLASSICAL NEAR 2n-GONS

As usual $\Delta_i(x) = \{y : d(x,y) = i\}$. Let $P = \mathcal{P}_1 \cup \mathcal{P}_2$ be the partition of P as the connected components of Δ_2. If $x,y \in \mathcal{P}_i$ and $d(x,y) = 2$, then there is a unique quad on x and y which we denote by $Q(x,y)$. Let $\mathcal{Q}$ be the collection of quads.

6.A. In this subsection we assume n = 3 and show conclusion (i) if (5.8) holds

(6.1) LEMMA. *Suppose* $Q_1,Q_2 \in \mathcal{Q}$, $Q_1 \neq Q_2$ *and* $Q_1 \cap Q_2 \neq \emptyset$. *Then* $Q_1 \cap Q_2 \in \Delta$.

PROOF. Let $x \in Q_1 \cap Q_2$. Suppose $x \in \mathcal{P}_1$. Choose $u_i \in Q_i \cap \Delta_2(x) = Q_i \cap \mathcal{P}_1$. Then $d(u_1,u_2) = 2$. Set $Q = Q(u_1,u_2)$. Now $x \notin Q$ for otherwise $Q = Q_1 = Q_2$. Therefore, the unique point $v \in Q$ with $d(v,x) = d(Q,z)$ is in $\mathcal{P}_2$ and $d(v,x) = 1$. Then $v \in x^{\perp} \cap u_i^{\perp} \subseteq Q_i$ and $\{x,v\} \in \Delta$. If $Q_1 \cap Q_2 \supsetneq \{x,v\}$, then either $|Q_1 \cap Q_2 \cap \mathcal{P}_1| > 1$ or $|Q_1 \cap Q_2 \cap \mathcal{P}_2| > 1$. In either case we get $Q_1 = Q_2$, a contradiction.

We shall for the remainder of this subsection say two distinct quads are "collinear" if they meet. If Q_1,Q_2 are collinear, let $\lambda(Q_1,Q_2) = \{Q \in \mathcal{Q} : Q \supseteq Q_1 \cap Q_2\}$. Let $\Lambda = \{\lambda(Q_1,Q_2) : Q_1 \neq Q_2 \in \mathcal{Q}, Q_1 \cap Q_2 \neq \emptyset\}$. We immediately have

(6.2) <u>LEMMA</u>. $(\mathcal{Q},\Lambda)$ *is a partial linear space.*

Note that lines are in one-to-one correspondence with edges in Δ. For such an edge, $\{x,a\}$, we will write $\lambda\{x,a\}$ for the corresponding line. The next lemma gives a concrete description of this line.

(6.3) <u>LEMMA</u>. *In* $\{x,a\} \in \Delta$, $\lambda\{x,a\} = \{Q(x,y)\ y \in \Delta(a)-\{x\}\}$

<u>PROOF</u>. If $y \in \Delta(a), y \neq x$, then $Q(x,y) \supseteq \{x,a\}$ and $Q(x,y) \in \lambda\{x,a\}$. On the other hand, if $Q \in \lambda\{x,a\}$, then for any $y \in Q \cap \Delta_2(x)$, $y \in \Delta(a)$ and $Q = Q(x,y)$.

(6.4) <u>PROPOSITION</u>. $(\mathcal{Q},\Lambda)$ *is a polar space of type* D_3.

<u>PROOF</u>. First we show $(\mathcal{Q},\Lambda)$ is a gamma space : let $\lambda = \lambda\{x,a\}$ for $\{x,a\} \in \Delta$ and $Q \in \mathcal{Q}$. If $Q \cap \{x,a\} \neq \emptyset$, then Q is collinear with each point of λ so we may assume $Q \cap \{x,a\} = \emptyset$. We show in this case Q is collinear with at most one point of λ . Suppose $Q \in \lambda$, $Q \cap Q_1 \neq \emptyset$. Let $Q \cap Q_1 = \{y,b\}$ where $\{a,y\}$, $\{b,x\} \in \Delta$. Suppose that $Q_1 \neq Q_2 \in \lambda$. Then $y \notin Q_2$, but $a \in Q_2 \cap \Delta(y)$. If $Q \cap Q_2 \neq \emptyset$, then $Q_2 \cap \Delta(y) \in Q$. Since $a \in Q$ we cannot have $Q \cap Q_2 \neq \emptyset$ as asserted. Thus $(\mathcal{Q},\Lambda)$ is a gamma space. Now consider a line $\lambda = \lambda\{x,a\}$ and a point $Q \in \mathcal{Q} \setminus \lambda$. Since diam $(P,\Gamma) = 3$, $Q \cap \Delta(a) \neq \emptyset$. By (6.3) this implies Q is collinear with some point of L and consequently $(\mathcal{Q},\Lambda)$ is a polar space. Since the induced structure on the lines of $(\mathcal{Q},\Lambda)$ contains a fixed Q is isomorphic to the dual of Q it follows from TITS [5] $(\mathcal{Q},\Gamma) \cong D_3(K)$, K a

Now it is obvious to see that for $x \in P$, $\hat{x} = \{Q \in \mathcal{Q} : x \in Q\}$ is a maximal singular subspace of the polar space $(\mathcal{Q},\Lambda)$. The result in this case follows.

6.B. Hence-forth assume $n \geq 4$. Set $\mathcal{P} = \mathcal{P}_1$ and $\Gamma = \Delta_2 | \mathcal{P}$.

(6.5) <u>NOTATION</u>. If $x,y \in \mathcal{P}$, $d(x,y) = 2$(so $d_\Gamma(x,y) = 1$), set $xy = Q(x,y) \cap \mathcal{P}$. Set $\mathcal{L} = \{xy : x,y \in \mathcal{P}, d_\Gamma(x,-) = 1\}$. For $x \in \mathcal{P}$, $x^* = \Gamma(x) \cup \{x\}$.

(6.6) <u>LEMMA</u>. $(\mathcal{P},\mathcal{Q})$ *is a strong* Γ*-space.*

<u>PROOF</u>. Let $x,y,z \in \mathcal{P}$ with $y \in \Gamma(x)$, $x,y \in \Gamma_d(z)$. Set $Q = Q(x,y)$. Let $a \in Q$

$d_\Delta(z,Q) = d_\Delta(z,a)$. If $a \in P$, then $d_\Delta(z,x) - 2 = 2d - 2$. In this case $\{a\} = \ell \cap \Gamma_{d-1}(z)$. If $a \in P_2$, then $d_\Delta(z,a) = 2d - 1$ and $xy = P \cap Q = P \cap \Delta(a) \subseteq \Delta_{2d}(z) = \Gamma_d(z)$, and so in this case $xy \subseteq \Gamma_d(z)$.

(6.7) LEMMA. *Let* $\ell \in L$, $x \in P$ *and* $\ell \subseteq \Gamma_d(x)$ *with* $d \geq 2$. *Then* $\ell^* \cap \Gamma_{d-1}(x)$ *is a non-empty singular subspace of* $(P,)$. $(\ell^* = \bigcap_{y \in \ell} y^*)$.

PROOF. Note, if $a \in P_2$, then $\Delta(a)$ is a singular subspace of (P,L). By definition of quads, there is a unique $Q \in L$, $Q \supseteq \ell$, which we denote by $Q(\ell)$. Let $a \in Q$ such that $d_\Delta(x,a) = d_\Delta(x,Q)$. Since $\ell \subseteq \Gamma_d(x) = \Delta_{2d}(x)$, $a \in P_2$. Therefore $d_\Delta(a,x) = 2d - 1$. Choose $y \in \Delta(a) \cap \Delta_{2d-2}(x)$. Then $y \in \ell^*$ since $y, \ell \subseteq \Delta(a)$. Also $y \in \Gamma_{d-1}(x)$, so $\Gamma_{d-1}(x) \cap \ell^* \neq \emptyset$.

We next show for any $y \in \Gamma_{d-1}(x) \cap \ell^*$ that $y \in \Delta(a)$ which will prove $\Gamma_{d-1}(x) \cap \ell^*$ is a Γ-clique by our first remark. Let $u,v \in \ell$. Consider $Q(y,u)$. Now $\Delta(y) \cap \Delta(u) \subseteq \Delta_{2d-1}(x)$. If $v \in Q(y,u)$, then $Q(y,u) = Q(u,v) = Q(\ell)$ contradicting $d_\Delta(x,y) = 2d - 2$ and $Q \cap P \subseteq \Gamma_d(x)$. Therefore $d_\Delta(Q(y,u),v) \geq 1$. But $d_\Delta(y,v) = d_\Delta(y,u) = 2$ and so it follows that if b is the unique point of $Q(x,u)$ closest to v, then $b \in P_2$ and $d_\Delta(b,v) = 1$.
Since $b \in \Delta(y)$, $d_\Delta(b,x) \leq 2d - 1$. Since $b \in \Delta(u) \cap \Delta(v)$, $b \in Q(u,v) = Q$. But $Q \cap \Delta_{2d-1}(x) = \{a\}$, so $b = a$. Since (P,Q) is a strong Γ-space $\Gamma_{d-1}(x) \cap \ell^*$ is a subspace and the lemma is proved.

(6.8) LEMMA. *Let* $x,y \in P$, $d_\Gamma(x,y) = 2, z \in \Gamma(x) \cap \Gamma(y)$. *Then there exists* $v \in \Gamma(x) \cap \Gamma(y) \cap \Gamma_2(z)$.

PROOF. Let $a \in P_2 \cap Q(x,z)$, $b \in P_2 \cap Q(y,z)$. As $d_\Gamma(x,y) = 2$, $a \neq b$. Since $z \in \Delta(a) \cap \Delta(b)$ we have $d_\Delta(a,b) = 2$ and $z \in Q(a,b)$. Let $u \in Q(a,b) \cap P$, $u \neq z$. $z \notin Q(x,y) \cap Q(y,u)$. For if $z \in Q(x,y) \cap Q(y,u)$, then $Q(x,y) = Q(z,u) = Q(y,u)$. Thus $d_\Gamma(x,y) = 1$, a contradiction. Now $P \cap Q(x,y)$, $P \cap Q(y,u) \subseteq \Gamma(z)$. It follows that there is a unique a_1, b_1 in $Q(x,u) \cap \Delta(z)$, $Q(y,u) \cap \Delta(z)$, respectively, namely a and b. Let $a_2 \in \Delta(x) \cap \Delta(u)$, $a_2 \neq a$ and b_2 choosen similarly. Then $a_2, b_2 \in \Delta_3(z)$. Then $Q(a_2,b_2) \cap \Gamma(z) = \{u\}$. Now if $v \in \Delta(a_2) \cap \Delta(b_2)$, $v \neq u$, then $v \in \Gamma(x) \cap \Gamma(y) \cap \Gamma_2(z)$.

(6.9) LEMMA. *Let* $\ell \in Q$, $x \in P$ *with* $\ell \subseteq \Gamma_2(x)$. *Then* $C(x,\ell) = x^* \cap \ell^*$ *properly contains a line.*

PROOF. Set $Q = Q(\ell)$. Let a be the unique point in $Q \cap \Delta_3(x)$. Let x,b,y, a be a geodesic from x to a. Then $Q(a,b) \cap P$ is a line contained in $C(x,\ell)$. Now let $c \in Q(x,y) \cap P_2$, $c \neq b$. Then $y \in Q(a,c)$ and $Q(a,c) \neq Q(a,b)$. Therefore $P \cap Q(q,c) \cap Q(a,b) = \{y\}$. But $P \cap Q(a,c)$ is a line in $C(x,\ell)$ and $P \cap Q(a,c) \neq P \cap Q(a,b)$ and (6.9) is proved.

(6.10) LEMMA. *If* $x,y \in P$, $d_\Gamma(x,y) = 2$, *then* $\Gamma(x) \cap \Gamma(y)$ *is a polar space of rank three.*

PROOF. $\Gamma(x) \cap \Gamma(y)$ is a Γ-space with thick lines. By (6.8) $\Gamma(x) \cap \Gamma(y)$ is non-degenerate. From (6.7) it follows that $\Gamma(x) \cap \Gamma(y)$ is a polar space.

Now let $z \in \Gamma(x) \cap \Gamma(y)$, $u \in \Gamma(x) \cap \Gamma(y) \cap \Gamma(z)$. Since $xz \subseteq \Gamma(u)$, $yz \subseteq \Gamma(u)$, there is a unique $b \in Q(x,z) \cap \Delta(u)$ and a unique $c \in Q(y,z) \cap \Delta(u)$. Then $u \in P \cap Q(b,c)$. It follows that the lines on z in $\Gamma(x) \cap \Gamma(y)$ is a grid isomorphic to $[Q(y,z) \cap P_2] \times [Q(x,z) \cap P_2]$. From this it follows that maximal singular subspaces of $\Gamma(x) \cap \Gamma(y)$ are planes and $\mathrm{rk}(\Gamma(x) \cap \Gamma(y)) = 3$.

We have now shown that (D1)-(D3) hold for (P,L). Thus, either $(P,L) \cong D_{n,n}(K)$ for some field K or (P,L) is a polar space of rank 4. However, in the latter case, by the end of 6.10 and TITS [5] we have $(P,L) \cong D_4(K) \cong D_{4,4}(K)$. Now the points in P_2 can be identified with the maximal singular subspaces of (P,L) with projective dimension n-1. From this identification it now follows that $P_1 \cup P_2$ can be identified with the maximal singular subspaces of an orthogonal space V of dimension $2n(\geq 8)$ over a field K, with index n, such that two are collinear if and only if they meet in an (n-1) dimensional subspace. This completes the proof of (5.10).

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